

# Planar Projections and Second Intrinsic Volume

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**ABSTRACT.** Consider random shadows of a cube and of a regular tetrahedron. Area and perimeter of the former are positively dependent (with correlation 0.915...), whereas area and perimeter of the latter appear to be negatively dependent. This is only one result of many, all involving generalizations of mean width.

Let  $C$  be a convex body in  $\mathbb{R}^n$ . Let  $S$  be a 1-dimensional subspace passing through the origin. A **width** is the length of the orthogonal projection of  $C$  on  $S$ . If  $S$  is uniformly distributed on the Grassmannian manifold  $\mathbb{G}^{n,1}$ , then the width  $w(S)$  is a random variable. First and second moments of  $w$  are known for  $C =$  the regular  $n$ -simplex,  $C =$  the  $n$ -cube and  $C =$  the regular  $n$ -crosspolytope, each centered at the origin [1, 2]. The **mean width** is sometimes called the **mean linear projection** [3]—a slight abuse of language — because a projection of  $C$  on  $S$  has only one parameter of interest: its length.

Let instead  $S$  be a 2-dimensional subspace passing through the origin. Orthogonal projections of  $C$  on  $S$  now have three parameters of interest. One parameter is the number of polygonal vertices; associated probabilities will be mentioned later. The other two parameters constitute direct generalizations of width and will be our main focus:

- a **chorowidth** is the polygonal area of the orthogonal projection of  $C$  on  $S$  (“choro” is Greek for place or area);
- a **periwidth** is the polygonal circumference of the orthogonal projection of  $C$  on  $S$  (“peri” is Greek for around or enclosing).

If  $S$  is uniformly distributed on  $\mathbb{G}^{n,2}$ , then chorowidth  $cw(S)$  and periwidth  $pw(S)$  are random variables. We shall examine joint moments of  $(cw, pw)$  for the same regular polytopes as before, for the special cases  $n = 3$  and  $n = 4$ .

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# 1. METHOD FOR COMPUTING MOMENTS

To generate a random 2-subspace  $S$  in  $\mathbb{R}^3$ , we select a random point  $U$  uniformly on the 2-sphere of unit radius. The desired subspace is the set of all vectors orthogonal to  $U$ .

We then project the (fixed) convex polyhedron  $C$  orthogonally onto  $S$ . This is done by forming the convex hull of images of all vertices of  $C$ . If  $C$  = the regular 3-simplex, the resultant polygon in the plane has 3 or 4 vertices almost surely. If  $C$  = the 3-cube, there are 6 vertices almost surely; if  $C$  = the regular 3-crosspolytope, there are 4 or 6 vertices almost surely. We compute the area and circumference of this polygon via standard formulas.

More precisely, if  $U = (x, y, z)$  is of unit length, then the matrix

$$M_3 = \begin{pmatrix} \sqrt{1-x^2} & -\frac{xy}{\sqrt{1-x^2}} & -\frac{xz}{\sqrt{1-x^2}} \\ 0 & \frac{z}{\sqrt{1-x^2}} & -\frac{y}{\sqrt{1-x^2}} \\ 0 & 0 & 0 \end{pmatrix}$$

projects  $C$  orthogonally onto a plane, rotated in  $\mathbb{R}^3$  to coincide with the 2-subspace spanned by  $(1, 0, 0)$  and  $(0, 1, 0)$  for convenience. Let  $T$  be the 3-row matrix whose columns constitute all vertices of  $C$ . Then the first 2 rows of  $M_3 T$  constitute all images of the vertices in  $\mathbb{R}^2$  and 2-dimensional convex hull algorithms apply naturally.

Spherical coordinates in  $\mathbb{R}^3$ :

$$x = \cos \theta \sin \varphi, \quad y = \sin \theta \sin \varphi, \quad z = \cos \varphi$$

will be used throughout for  $U$ , where  $0 \leq \theta < 2\pi$ ,  $0 \leq \varphi \leq \pi$ . The corresponding Jacobian determinant is  $\sin \varphi$ ; it is best to think of  $(\theta, \varphi)$  as possessing joint density  $\frac{1}{4\pi} \sin \varphi$ .

Moving up a dimension, to generate a random 2-subspace  $S$  in  $\mathbb{R}^4$ , we first select a random point  $U$  uniformly on the 3-sphere of unit radius. The set of unit vectors orthogonal to  $U$  form a 2-sphere; we select a random point  $V$  uniformly on this 2-sphere. The desired subspace is then the set of all vectors orthogonal to both  $U$  and  $V$ .

We next project the (fixed) convex polyhedron  $C$  orthogonally onto  $S$ . This is done by forming the convex hull of images of all vertices of  $C$ . If  $C$  = the regular 4-simplex, the resultant polygon in the plane has 3, 4 or 5 vertices almost surely. If  $C$  = the 4-cube, there are 8 vertices almost surely; if  $C$  = the regular 4-crosspolytope, there are 4, 6 or 8 vertices almost surely. Area and circumference of this polygon are computed as before.

More precisely, if  $U = (x, y, z, w)$  and  $V = (p, q, r, s)$  are orthogonal and of unit

length, then the matrix

$$M_4 = \begin{pmatrix} \sqrt{1-p^2-x^2} & -\frac{pq+xy}{\sqrt{1-p^2-x^2}} & -\frac{pr+xz}{\sqrt{1-p^2-x^2}} & -\frac{ps+xw}{\sqrt{1-p^2-x^2}} \\ 0 & \frac{rw-sz}{\sqrt{1-p^2-x^2}} & -\frac{qw-sy}{\sqrt{1-p^2-x^2}} & \frac{qz-ry}{\sqrt{1-p^2-x^2}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

projects  $C$  orthogonally onto a plane, rotated in  $\mathbb{R}^4$  to coincide with the 2-subspace spanned by  $(1, 0, 0, 0)$  and  $(0, 1, 0, 0)$  for convenience. Let  $T$  be the 4-row matrix whose columns constitute all vertices of  $C$ . Then the first 2 rows of  $M_4T$  constitute all images of the vertices in  $\mathbb{R}^2$  and 2-dimensional convex hull algorithms again apply naturally.

Spherical coordinates in  $\mathbb{R}^4$ :

$$x = \cos \theta \sin \varphi \sin \psi, \quad y = \sin \theta \sin \varphi \sin \psi, \quad z = \cos \varphi \sin \psi, \quad w = \cos \psi$$

will be used throughout for  $U$ , where  $0 \leq \theta < 2\pi$ ,  $0 \leq \varphi \leq \pi$ ,  $0 \leq \psi \leq \pi$ . It follows that  $V$  is given by

$$\begin{pmatrix} p \\ q \\ r \\ s \end{pmatrix} = \cos \kappa \sin \lambda \begin{pmatrix} -y \\ x \\ -w \\ z \end{pmatrix} + \sin \kappa \sin \lambda \begin{pmatrix} -z \\ w \\ x \\ -y \end{pmatrix} + \cos \lambda \begin{pmatrix} -w \\ -z \\ y \\ x \end{pmatrix}$$

where  $0 \leq \kappa < 2\pi$ ,  $0 \leq \lambda \leq \pi$ . It is best to think of  $(\theta, \varphi, \psi, \kappa, \lambda)$  as possessing joint density  $\frac{1}{2\pi^2} \sin \varphi \sin^2 \psi \frac{1}{4\pi} \sin \lambda$ .

## 2. THREE-DIMENSIONAL RESULTS

From the foregoing, we calculate mean chorowidth and mean periwidth via integration:

$$\mathbb{E}(cw) = \int_0^{2\pi} \int_0^\pi \text{area}(\text{convex\_hull}(M_3T)) \frac{1}{4\pi} \sin \varphi d\varphi d\theta,$$

$$\mathbb{E}(pw) = \int_0^{2\pi} \int_0^\pi \text{circumference}(\text{convex\_hull}(M_3T)) \frac{1}{4\pi} \sin \varphi d\varphi d\theta$$

and likewise for mean square chorowidth, mean square periwidth and joint moment.

### 2.1. 3-Simplex (Tetrahedron).

$$\mathbb{E}(cw) = \frac{\sqrt{3}}{4} = 0.433012701892219... = \frac{\text{surface area}}{4},$$

$$\mathbb{E}(cw^2) = \frac{1}{8} + \frac{\sqrt{2}}{4\pi} - \frac{1}{8\pi} \operatorname{arcsec}(3) = 0.188561220515812...,$$

$$\mathbb{E}(pw) = \frac{3}{2}(\pi - \operatorname{arcsec}(3)) = 2.865949854373527...,$$

$$\mathbb{E}(pw^2) = 8.2170808733...,$$

$$\mathbb{E}(cw \cdot pw) = 1.2406348222...$$

which imply that the correlation between  $cw$  and  $pw$  is  $\approx -0.188$ . Closed-form expressions for  $\mathbb{E}(pw^2)$  and  $\mathbb{E}(cw \cdot pw)$  are not known. The fact that

$$\left( \begin{array}{c} \text{surface area of a} \\ \text{3D convex body} \end{array} \right) = 4 \left( \begin{array}{c} \text{mean area of} \\ \text{its 2D shadow} \end{array} \right)$$

was first noticed by Cauchy [4, 5, 6, 7]. We also have [8, 9]

$$\mathbb{P}(\text{projection has 3 vertices}) = \frac{2}{\pi} (3 \operatorname{arcsec}(3) - \pi) = 0.3509593121...,$$

$$\mathbb{P}(\text{projection has 4 vertices}) = \frac{3}{\pi} (\pi - 2 \operatorname{arcsec}(3)) = 0.6490406878...$$

### 2.2. 3-Cube.

$$\mathbb{E}(cw) = \frac{3}{2} = 1.5 = \frac{\text{surface area}}{4},$$

$$\mathbb{E}(cw^2) = 1 + \frac{4}{\pi} = 2.273239544735162...,$$

$$\mathbb{E}(pw) = \frac{3}{2}\pi = 4.712388980384689...,$$

$$\mathbb{E}(pw^2) = 8 + 6\pi {}_3F_2\left(-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}; 1, 2; 1\right) = 22.237117433439470...,$$

$$\mathbb{E}(cw \cdot pw) = 2 + \frac{16}{\pi} = 7.092958178940650...$$

which imply that the correlation between  $cw$  and  $pw$  is 0.915.... We shall present more details underlying these results later, especially that involving the generalized hypergeometric function

$${}_3F_2(a_1, a_2, a_3; b_1, b_2; z) = \frac{\Gamma(b_1)\Gamma(b_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)} \sum_{k=0}^{\infty} \frac{\Gamma(a_1+k)\Gamma(a_2+k)\Gamma(a_3+k)}{\Gamma(b_1+k)\Gamma(b_2+k)} \frac{z^k}{k!}.$$

whose appearance is quite unexpected.

### 2.3. 3-Crosspolytope (Octahedron).

$$\mathbb{E}(cw) = \frac{\sqrt{3}}{2} = 0.866025403784438... = \frac{\text{surface area}}{4},$$

$$\mathbb{E}(cw^2) = \frac{1}{2} + \frac{\sqrt{2}}{\pi} - \frac{1}{2\pi} \operatorname{arcsec}(3) = 0.754244882063249...,$$

$$\mathbb{E}(pw) = 3 \operatorname{arcsec}(3) = 3.692878252022324...,$$

$$\mathbb{E}(pw^2) = 13.6639421274...,$$

$$\mathbb{E}(cw \cdot pw) = 3.2074623048...$$

which imply that the correlation between  $cw$  and  $pw$  is  $\approx 0.878$ . The fact that  $cw_{3\text{-crosspolytope}}$  behaves like  $2 \cdot cw_{3\text{-simplex}}$  will be discussed shortly. We also have [10]

$$\mathbb{P}(\text{projection has 4 vertices}) = \frac{3}{\pi} (\pi - 2 \operatorname{arcsec}(3)) = 0.6490406878...,$$

$$\mathbb{P}(\text{projection has 6 vertices}) = \frac{2}{\pi} (3 \operatorname{arcsec}(3) - \pi) = 0.3509593121....$$

See [11, 12, 13] for related probabilities governing planar cross sections of the tetrahedron, cube and octahedron (rather than projections).

## 3. TABLES OF INTRINSIC VOLUMES

Let  $\square$  be a rectangular 4-parallelepiped in  $\mathbb{R}^4$  of dimensions  $z_1, z_2, z_3, z_4$ . It is well-known that [14]

$$V_4(\square) = z_1 z_2 z_3 z_4,$$

$$V_3(\square) = z_1 z_2 z_3 + z_1 z_2 z_4 + z_1 z_3 z_4 + z_2 z_3 z_4,$$

$$V_2(\square) = z_1 z_2 + z_1 z_3 + z_1 z_4 + z_2 z_3 + z_2 z_4 + z_3 z_4,$$

$$V_1(\square) = z_1 + z_2 + z_3 + z_4$$

are the elementary symmetric polynomials in four variables. In  $\mathbb{R}^n$ , there are  $n$  such intrinsic volumes, corresponding to the  $n$  elementary symmetric polynomials. Limiting approximation arguments enable us to compute  $V_j(C)$  for arbitrary convex  $C$ .

One motivation for our work is to generalize the following two tables [4] for  $n = 2$ :

$$V_2 = \text{area}, \quad 2V_1 = \text{circumference}$$

and  $n = 3$ :

$$V_3 = \text{volume}, \quad 2V_2 = \text{surface area}, \quad \frac{1}{2}V_1 = \text{mean width}$$

to  $n = 4$ :

$$\begin{aligned} V_4 &= \text{hypervolume}, \\ 2V_3 &= \text{hyper-surface area}, \\ \frac{1}{3}V_2 &= \text{mean chorowidth}, \\ \frac{4}{3}V_1 &= \text{mean periwidth} = \pi (\text{mean width}). \end{aligned}$$

Another motivation is to use formulas for  $V_2$  in [15, 16] to deduce that

$$\mathbb{E}(cw_{n\text{-simplex}}) = \frac{n(n+1)}{8\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-3x^2} \left( \frac{1 + \text{erf}(x)}{2} \right)^{n-2} dx$$

for the regular  $n$ -simplex in  $\mathbb{R}^n$ ;

$$\mathbb{E}(cw_{n\text{-cube}}) = \frac{n}{2}$$

for the  $n$ -cube in  $\mathbb{R}^n$ ; and

$$\mathbb{E}(cw_{n\text{-crosspolytope}}) = \frac{n(n-2)}{\sqrt{\pi}} \int_0^{\infty} e^{-3x^2} \text{erf}(x)^{n-3} dx$$

for the regular  $n$ -crosspolytope in  $\mathbb{R}^n$ . In contrast,  $\mathbb{E}(pw_n)$  are obtained simply by forming the product  $\pi \cdot \mathbb{E}(w_n)$ , and associated  $\mathbb{E}(w_n)$  are tabulated in [1, 2]. No such general expressions are available for higher moments.

#### 4. FOUR-DIMENSIONAL RESULTS

From the foregoing, we calculate mean chorowidth and mean periwidth via integration:

$$\mathbb{E}(cw) = \int_0^{2\pi} \int_0^\pi \int_0^{2\pi} \int_0^\pi \int_0^\pi \text{area}(\text{convex\_hull}(M_4T)) \frac{1}{8\pi^3} \sin \varphi \sin^2 \psi \sin \lambda d\psi d\varphi d\theta d\lambda d\kappa,$$

$$\mathbb{E}(pw) = \int_0^{2\pi} \int_0^\pi \int_0^{2\pi} \int_0^\pi \int_0^\pi \text{circumference}(\text{convex\_hull}(M_4T)) \frac{1}{8\pi^3} \sin \varphi \sin^2 \psi \sin \lambda d\psi d\varphi d\theta d\lambda d\kappa$$

and likewise for mean square chorowidth, mean square periwidth and joint moment.

##### 4.1. 4-Simplex.

$$\mathbb{E}(cw) = \frac{5\sqrt{3}}{12\pi} (\pi - \text{arcsec}(4)) = 0.418889720727840...,$$

$$\mathbb{E}(cw^2) = 0.176...,$$

$$\mathbb{E}(pw) = \frac{10}{3\pi} (2\pi - 3 \text{arcsec}(3)) = 2.748401146360593...,$$

$$\mathbb{E}(pw^2) = 7.56...,$$

$$\mathbb{E}(cw \cdot pw) = 1.15...$$

which imply that the correlation between  $cw$  and  $pw$  is  $\approx 0.1$ . Closed-form expressions for  $\mathbb{E}(cw^2)$ ,  $\mathbb{E}(pw^2)$  and  $\mathbb{E}(cw \cdot pw)$  are not known. We also have

$$\mathbb{P}(\text{projection has 3 vertices}) \approx 0.146,$$

$$\mathbb{P}(\text{projection has 4 vertices}) \approx 0.585,$$

$$\mathbb{P}(\text{projection has 5 vertices}) \approx 0.269.$$

These values are consistent with a theorem in [8] that the expected number of vertices should be

$$20\sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} e^{-2x^2} \left( \frac{1 + \text{erf}(x)}{2} \right)^3 dx = 10 \left( 1 - \frac{3}{2\pi} \text{arcsec}(3) \right) = 4.122...$$

$$\approx 3(0.146) + 4(0.585) + 5(0.269).$$

#### 4.2. 4-Cube.

$$\mathbb{E}(cw) = 2,$$

$$\mathbb{E}(cw^2) = 4.04\dots,$$

$$\mathbb{E}(pw) = \frac{16}{3} = 5.\overline{3},$$

$$\mathbb{E}(pw^2) = 28.4\dots,$$

$$\mathbb{E}(cw \cdot pw) = 10.7\dots$$

which imply that the correlation between  $cw$  and  $pw$  is  $\approx 0.9$ .

#### 4.3. 4-Crosspolytope.

$$\mathbb{E}(cw) = \frac{4\sqrt{3}}{9} = 0.769800358919501\dots,$$

$$\mathbb{E}(cw^2) = 0.598\dots,$$

$$\mathbb{E}(pw) = \frac{16}{\pi} (\pi - 2 \operatorname{arcsec}(3)) = 3.461550335020567\dots,$$

$$\mathbb{E}(pw^2) = 12.0\dots,$$

$$\mathbb{E}(cw \cdot pw) = 2.67\dots$$

which imply that the correlation between  $cw$  and  $pw$  is  $\approx 0.8$ . We also have

$$\mathbb{P}(\text{projection has 4 vertices}) \approx 0.463,$$

$$\mathbb{P}(\text{projection has 6 vertices}) \approx 0.478,$$

$$\mathbb{P}(\text{projection has 8 vertices}) \approx 0.059.$$

These values are consistent with a theorem in [10] that the expected number of vertices should be

$$\begin{aligned} 48\sqrt{\frac{2}{\pi}} \int_0^\infty e^{-2x^2} \operatorname{erf}(x)^2 dx &= 24 \left( 1 - \frac{2}{\pi} \operatorname{arcsec}(3) \right) = 5.192\dots \\ &\approx 4(0.463) + 6(0.478) + 8(0.059). \end{aligned}$$



### 5. FURTHER WORK

After having written the preceding, we discovered [17], which provides new insights in the 3-dimensional case. Earlier papers in this line of thought include [18, 19, 20, 21]. Consider the 3-cube with vertices

$$T = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}.$$

A starting point here is the simultaneous system of equations

$$\begin{cases} cw = x + y + z \\ pw = 2 \left( \sqrt{1-x^2} + \sqrt{1-y^2} + \sqrt{1-z^2} \right) \end{cases}$$

given a unit vector  $U = (x, y, z)$  in the first octant. This makes possible, for example, the derivation of a closed-form marginal density for  $cw$  (although not a joint density). Let us focus on computing  $\mathbb{E}(pw^2)$  and  $\mathbb{E}(cw \cdot pw)$ . Contributing to  $\mathbb{E}(pw^2)$  are three terms like

$$I = \int_0^{\pi/2} \int_0^{\pi/2} (1 - \cos^2 \theta \sin^2 \varphi) \sin \varphi \, d\theta \, d\varphi = \frac{\pi}{3}$$

and six terms like

$$\begin{aligned} J &= \int_0^{\pi/2} \int_0^{\pi/2} \sqrt{1 - \sin^2 \theta \sin^2 \varphi} \sqrt{1 - \cos^2 \varphi} \sin \varphi \, d\theta \, d\varphi \\ &= \int_0^{\pi/2} E(\sin \varphi) \sin^2 \varphi \, d\varphi = \frac{\pi^2}{8} {}_3F_2 \left( -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}; 1, 2; 1 \right) \end{aligned}$$

where

$$E(\xi) = \int_0^{\pi/2} \sqrt{1 - \xi^2 \sin^2(\theta)} \, d\theta = \int_0^1 \sqrt{\frac{1 - \xi^2 t^2}{1 - t^2}} \, dt$$

is the complete elliptic integral of the second kind. The final result is  $32(3I+6J)/(4\pi)$ . For  $\mathbb{E}(cw \cdot pw)$ , the calculations are simpler, with  $I = \pi/6$  and  $J = 2/3$ .

An analogous system of equations for the regular 3-simplex with vertices

$$T = \begin{pmatrix} 0 & \frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{6} & -\frac{\sqrt{3}}{6} \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{\sqrt{6}}{4} & -\frac{\sqrt{6}}{12} & -\frac{\sqrt{6}}{12} & -\frac{\sqrt{6}}{12} \end{pmatrix}$$

is more difficult. Define constants

$$\gamma = \operatorname{arccot}(2\sqrt{2}) = 0.339\dots, \quad \delta = \operatorname{arccot}(\sqrt{2}) = 0.615\dots$$

and functions

$$\alpha(\varphi) = \begin{cases} \operatorname{arcsec}(2\sqrt{2}\tan\varphi) & \text{if } \gamma \leq \varphi \leq \pi/2, \\ 0 & \text{if } 0 \leq \varphi \leq \gamma, \end{cases} \quad \beta(\varphi) = \frac{2\pi}{3} - \alpha(\varphi).$$

Given a unit vector  $U = (x, y, z)$  in the first dodecant (one of twelve regions in 3-space), we have

$$cw = \begin{cases} \frac{1}{12}(\sqrt{6}x + 3\sqrt{2}y + \sqrt{3}z) & \text{if } \delta \leq \varphi \leq \frac{\pi}{2} \text{ and } \beta(\varphi) \leq \theta \leq \frac{\pi}{3}, \\ \frac{\sqrt{3}}{6}(\sqrt{2}x + z) & \text{if } \gamma \leq \varphi \leq \frac{\pi}{2} \text{ and } 0 \leq \theta \leq \min\{\alpha(\varphi), \beta(\varphi)\}, \\ \frac{\sqrt{3}}{4}z & \text{if } 0 \leq \varphi \leq \delta \text{ and } \alpha(\varphi) \leq \theta \leq \frac{\pi}{3} \end{cases}$$

and the three respective expressions for  $pw$  are

$$\frac{1}{6} \left( 3\sqrt{4 - (\sqrt{3}x - y)^2} + \sqrt{3}\sqrt{12 - (x - \sqrt{3}y + 2\sqrt{2}z)^2} + 2\sqrt{3}\sqrt{3 - (x - \sqrt{2}z)^2} \right),$$

$$\begin{aligned} & \frac{\sqrt{3}}{6} \left( \sqrt{3}\sqrt{4 - (\sqrt{3}x - y)^2} + \sqrt{3}\sqrt{4 - (\sqrt{3}x + y)^2} + \right. \\ & \left. \sqrt{12 - (x - \sqrt{3}y + 2\sqrt{2}z)^2} + \sqrt{12 - (x + \sqrt{3}y + 2\sqrt{2}z)^2} \right), \end{aligned}$$

$$\frac{1}{2} \left( \sqrt{4 - (\sqrt{3}x - y)^2} + 2\sqrt{1 - y^2} + \sqrt{4 - (\sqrt{3}x + y)^2} \right).$$

An analogous system of equations for the regular 3-crosspolytope with vertices

$$T = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$$

is similar. Define functions

$$\alpha(\varphi) = \begin{cases} \pi/4 - \operatorname{arcsec}(\sqrt{2}\tan\varphi) & \text{if } \delta \leq \varphi \leq \pi/2, \\ \pi/4 & \text{if } 0 \leq \varphi \leq \delta, \end{cases} \quad \beta(\varphi) = -\alpha(\varphi)$$

where the constant  $\delta$  is the same as before. Given a unit vector  $U = (x, y, z)$  in the first hexadecant (one of sixteen regions in 3-space), we have

$$cw = \begin{cases} \frac{1}{2}(x + y + z) & \text{if } \delta \leq \varphi \leq \frac{\pi}{2} \text{ and } \max\{\alpha(\varphi), \beta(\varphi)\} \leq \theta \leq \frac{\pi}{4}, \\ x & \text{if } \frac{\pi}{4} \leq \varphi \leq \frac{\pi}{2} \text{ and } 0 \leq \theta \leq \beta(\varphi), \\ z & \text{if } 0 \leq \varphi \leq \frac{\pi}{4} \text{ and } 0 \leq \theta \leq \alpha(\varphi) \end{cases}$$

and the three respective expressions for  $pw$  are

$$\begin{aligned} & \sqrt{2} \left( \sqrt{2 - (x + y)^2} + \sqrt{2 - (x + z)^2} + \sqrt{2 - (y + z)^2} \right), \\ & \sqrt{2} \left( \sqrt{2 - (y - z)^2} + \sqrt{2 - (y + z)^2} \right), \\ & \sqrt{2} \left( \sqrt{2 - (x - y)^2} + \sqrt{2 - (x + y)^2} \right). \end{aligned}$$

A rigorous proof that  $cw_{3\text{-crosspolytope}}$  and  $2 \cdot cw_{3\text{-simplex}}$  are identically distributed remains open.

Let us return finally to (ordinary) width  $w$  and questions left unanswered in [22]. The 2-cube (square) and regular 2-simplex (equilateral triangle) have vertices

$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \quad \begin{pmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{6} & -\frac{\sqrt{3}}{6} \end{pmatrix}$$

respectively. Given a unit vector  $(x, y)$  in the first quadrant,  $w = x + y$  for the square. Given a unit vector  $(x, y)$  in the first sextant,  $w = \frac{\sqrt{3}}{2}x + \frac{1}{2}y$  for the triangle. The corresponding width densities are

$$\begin{cases} \frac{4}{\pi} \frac{1}{\sqrt{2 - w^2}} & \text{if } 1 \leq w < \sqrt{2}, \\ 0 & \text{otherwise,} \end{cases} \quad \begin{cases} \frac{6}{\pi} \frac{1}{\sqrt{1 - w^2}} & \text{if } \frac{\sqrt{3}}{2} \leq w < 1, \\ 0 & \text{otherwise} \end{cases}$$

respectively, via simple argument.

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